$$\begin{cases} f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \\ f(x+2\pi) = f(x) \end{cases}$$

$$f(x+2L) = f(x)$$

$$\begin{array}{ccc} f & & g \\ \downarrow & & \downarrow \\ x \in [-L, L] & \longrightarrow & t \in [-\pi, \pi] \\ \downarrow & & \downarrow \\ & t = \dfrac{\pi}{L} x & \end{array}$$

$$\underbrace{g(t) = f(x)}_{t = \frac{\pi}{L} x}$$

$$t + 2\pi \longrightarrow \frac{\pi}{L} x + 2\pi = \frac{\pi}{L}(\underbrace{x+2L}_{x})$$

$$g(t+2\pi) = f(x+2L) = f(x) = g(t)$$

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

$\downarrow$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos nt \, dt = \frac{1}{\pi} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \cdot \frac{\pi}{L} dx$$

$$t = \frac{\pi}{L} x \qquad = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$dt = \pi/L \, dx$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

$$\begin{cases} f(x+2L)=f(x) \\ f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L} \\ a_0=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,dx,\quad a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,\quad b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases}$$

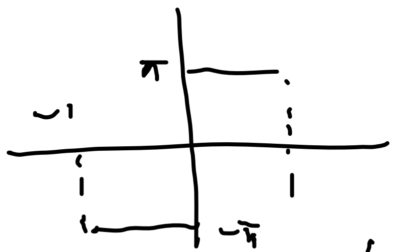
$n=1,2,\dots$

$$f(x)=\begin{cases} -\pi & -1<x<0 \\ \pi & 0<x<1 \end{cases}$$

فرد

$2L=2$

$L=1$



فرد $\begin{cases} a_0=0,\ a_n=0 \\ b_n=\dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \end{cases}$

$$b_n=\frac{2}{1}\int_0^1 \pi\sin\left(\frac{n\pi x}{1}\right)dx$$

$$=-\frac{2\pi}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$=-\frac{2}{n}(\cos n\pi-1)=\frac{2}{n}\left(1-(-1)^n\right)$$

(فرد n) $b_n=\frac{4}{n}$, (زوج n) $b_n=0$

$$b_n = \begin{cases} \frac{4}{n} & \text{ɔjn} \\ 0 & \text{ɛ,, } n \end{cases}$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$= 0 + 0 + \underbrace{\sum_{n=1}^{\infty}}_{(\text{ɔjn})} \frac{4}{n}\sin\left(\frac{n\pi x}{1}\right)$$

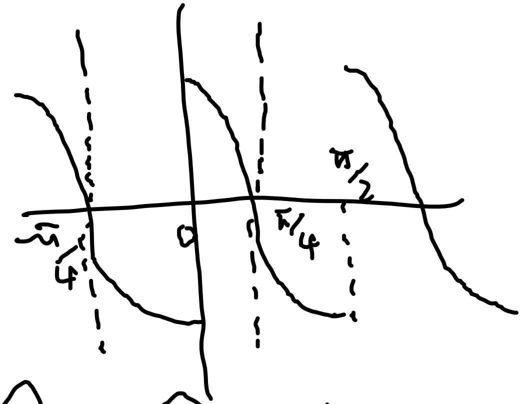
$$= 4\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x) = \begin{cases} -\pi & -1<x<0 \\ 0 & x=0 \\ \pi & 0<x<1 \end{cases}$$

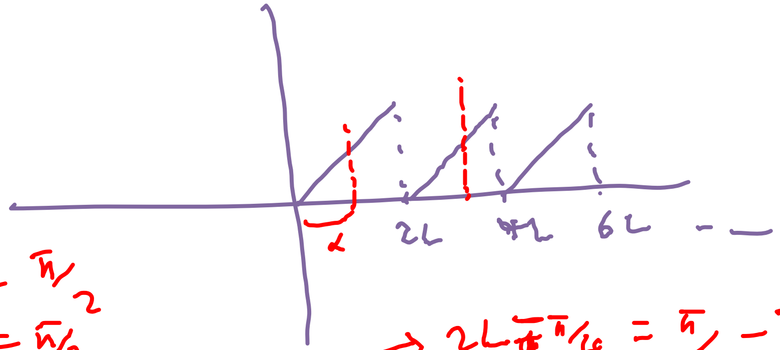
Example. $f(x) = \cos 2x, \quad 0 < x < \frac{\pi}{2}$

$$f\left(x+\frac{\pi}{2}\right) = f(x)$$

$$a_n = \frac{1}{L}\int_{\underbrace{-L+\alpha}_{-\pi/4}}^{\overset{\pi/4}{L+\alpha}} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$a_0 = 0$$





$$-\int_0^{2L} \longrightarrow \int_{0+\alpha}^{2L+\alpha}$$

$2L = \pi/2$

$L = \pi/4$

$\longrightarrow 2L - \pi/4 = \pi/2 - \pi/4 = \pi/4$

$$b_n = \frac{1}{L}\int_{-\pi/4}^{\pi/4} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{8}{\pi}\int_0^{\pi/4} \cos 2x \cdot \sin(4nx)\,dx$$

$0 - \pi/4 = -\pi/4$

$\pi/4$

$=$

$$\|f\|^2 = \int_a^b |f(x)|^2 \, dx$$

$$\|f-g\|^2 = \int_a^b |f(x)-g(x)|^2 \, dx$$

$$f(x) \longrightarrow \frac{a_0}{2} + \sum_{n=1}^{\infty \to N} a_n \cos n\bar{x} + b_n \sin n\bar{x}$$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx$$

مجموع جزئی N ام سری فوریه

خطای S_N ؟

$$E_N^* = \|f - S_N\|^2 \longrightarrow$$ خطای مجموع جزئی N ام سری فوریه

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos nx + B_n \sin nx$$

مقایسه

$$E_N = \|f - F\|^2$$

$$E_N = \|f - F\|^2 = \int_{-\pi}^{\pi} (f - F)^2 dx = \int_{-\pi}^{\pi} \{ f^2 - 2fF + F^2 \} dx$$

$$= \int_{-\pi}^{\pi} f^2 dx - 2 \underbrace{\int_{-\pi}^{\pi} fF dx}_{\mathrm{I}} + \underbrace{\int_{-\pi}^{\pi} F^2 dx}_{\mathrm{II}}$$

$$\int_{-\pi}^{\pi} f(x)F(x) dx = \int_{-\pi}^{\pi} \frac{A_0}{2} f(x) dx + \sum_{n=1}^{N} A_n \int_{-\pi}^{\pi} f(x)\cos nx \, dx + B_n \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{A_0}{2} \pi a_0 + \sum_{n=1}^{N} A_n \pi a_n + B_n \pi b_n$$

$$= \pi \left\{ \frac{A_0}{2} a_0 + \sum_{n=1}^{N} a_n A_n + b_n B_n \right\}$$

$$\int_{-\pi}^{\pi} F^2(x) dx = \pi \left\{ \frac{A_0^2}{2} + \sum_{n=1}^{N} A_n^2 + B_n^2 \right\}$$

$$E_N = \int_{-\pi}^{\pi} f^2(x)\,dx - 2\pi\left\{ \frac{A_0 a_0}{2} + \sum_{n=1}^{N} a_n A_n + b_n B_n \right\} + \pi\left\{ \frac{A_0^2}{2} + \sum_{n=1}^{N} A_n^2 + B_n^2 \right\}$$

$$E_N^* = \int_{-\pi}^{\pi} f^2(x)\,dx - 2\pi\left\{ \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \right\} + \pi\left\{ \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \right\}$$

$$= \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\left\{ \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \right\} \leftarrow$$ نتیجه 1

$$E_N - E_N^* = \pi\left\{ \frac{1}{2}(A_0 - a_0)^2 + \sum_{n=1}^{N} (A_n - a_n)^2 + (B_n - b_n)^2 \right\} \geq 0$$

$$E_N \geq E_N^* \longrightarrow$$ مجموع جزئی N ام سری فوریه

در بین تمام چندجمله‌ای‌های مثلثاتی درجه N کمترین خطا را دارد.

$E_N^* \longrightarrow$ square error (بهترین تقریب).

$$E_N^* = \min_{F} E_N$$

$$E_N^* \geq 0 \Rightarrow \int_{-\pi}^{\pi} f^2\,dx - \pi\left\{\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2\right\} \geq 0$$

N دلخواه

$N \to \infty$

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} f^2(x)\,dx}$$

نامساوی بسل

$\to \|f\|^2 < \infty$

$\downarrow$

square integrable

انتگرال پذیر مربعی

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx$$

اتحاد پارسوال

مثال. $f(x) = x + \pi \quad , \quad -\pi < x < \pi$

$g(x) = x$

$a_0 = 0 \,,\; a_n = 0$

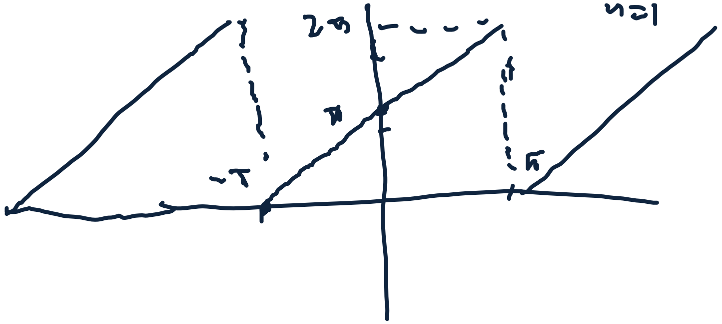
$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx$$

$$\begin{array}{c|c|l} & x & \sin nx \\ \hline + & 1 & -\frac{1}{n}\cos nx \\ - & 0 & -\frac{1}{n^2}\sin nx \end{array}$$

$$= \frac{2}{\pi}\left\{-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi}{n}\cos n\pi\right\} = \frac{2}{n}(-1)^{n+1}$$

$$f(x) \longrightarrow \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$



$$\frac{a_0}{2}$$

$$= \begin{cases} x+\pi & -\pi < x < \pi \\ \pi & x = \pm\pi \end{cases}$$

$f(x) = x + \pi$

$f(-\pi) = 0$

$f(\pi) = 2\pi$

$f(0) = \pi$

$$\int_{-\pi}^{\pi} f^2(x)\,dx = \int_{-\pi}^{\pi} (x+\pi)^2\,dx$$

$$= \frac{1}{3}(x+\pi)^3 \Big|_{-\pi}^{\pi} = \frac{8\pi^3}{3}$$

$$E_N^* = \int_{-\pi}^{\pi} f^2(x)\,dx - \pi\left\{\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2\right\} = \frac{8\pi^3}{3} - \pi\left\{\frac{(2\pi)^2}{2} + \sum_{n=1}^{N} 0^2 + \frac{4}{n^2}\right\}$$

$$= \frac{8\pi^3}{3} - 2\pi^3 - 4\sum_{n=1}^{N}\frac{1}{n^2} = \frac{2}{3}\pi^3 - 4\pi\sum_{n=1}^{N}\frac{1}{n^2}$$

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases} \quad , \quad f(x+2\pi) = f(x)$$

$$a_0 = a_n = 0, \quad b_n = \frac{2k}{n\pi}\left(1 - \cos n\pi\right) = \frac{2k}{n\pi}\left(1 - (-1)^n\right)$$

$$\int_{-\pi}^{\pi} f^2(x)\,dx = 2\int_0^{\pi} k^2\,dx = 2k^2\pi$$

$$b_n = \begin{cases} \dfrac{4k}{n\pi} & \text{فرد } n \\ 0 & \text{زوج } n \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

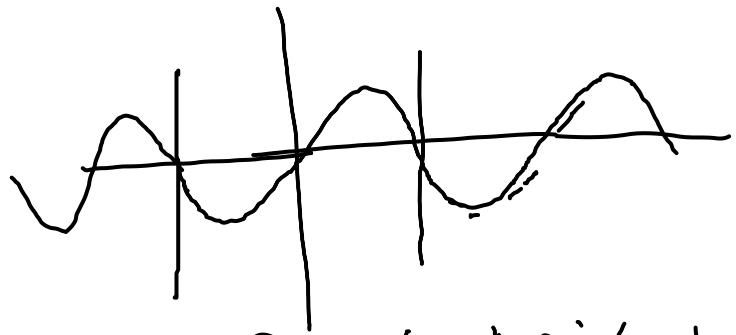
$$\frac{1}{\pi}\cdot 2k^2\pi = 0 + \sum_{\substack{n=1 \\ (\text{فرد } n)}}^{\infty} 0^2 + \left(\frac{4k}{n\pi}\right)^2$$

$$2k^2 = \sum_{\substack{n=1 \\ (\text{فرد } n)}}^{\infty} \frac{4^2 k^2}{n^2\pi^2} \implies \sum_{\substack{n=1 \\ (\text{فرد } n)}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{2\times 4} = \frac{\pi^2}{8}$$

$$f(x) = \begin{cases} x(x+\pi) & -\pi < x < 0 \\ x(-x+\pi) & 0 < x < \pi \end{cases}$$

$\rightarrow$ فرد

$$f(x+2\pi) = f(x)$$



$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} x(-x+\pi)\sin(nx)\,dx$$

$\oplus$	$x(-x+\pi)$	$\sin(nx)$
$\ominus$	$-2x+\pi$	$-\frac{1}{n}\cos nx$
$\oplus$	-2	$-\frac{1}{n^2}\sin nx$
	0	$\frac{1}{n^3}\cos nx$

$$= \frac{2}{\pi}\left\{ x(-x+\pi)\left(-\frac{1}{n}\cos nx\right) + (-2x+\pi)\frac{\sin nx}{n^2} - \frac{2}{n^3}\cos nx \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \cdot \frac{-2}{n^3}\left\{\cos n\pi - 1\right\} = \frac{4}{\pi n^3}\left(1-(-1)^n\right) = \begin{cases} \frac{8}{n^3\pi} & \text{فرد} \\ 0 & \text{زوج} \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_n \sin nx = \sum_{\substack{n=1 \\ (\text{فرد})}}^{\infty} \frac{8}{n^3\pi}\sin(nx)$$

$$\int_{-\pi}^{\pi} f^2(x)\,dx = 2\int_{0}^{\pi} f^2(x)\,dx = 2\int_{0}^{\pi} x^2(-x+\pi)^2\,dx$$

$$= 2\left\{ \frac{x^5}{5} - 2\pi\frac{x^4}{4} + \pi^2\frac{x^3}{3} \right\}_0^{\pi} = 2\left\{ \frac{\pi^5}{5} - \frac{\pi^5}{2} + \frac{\pi^5}{3} \right\}$$

$$= 2\cdot\frac{6-15+10}{30}\pi^5 = \frac{\pi^5}{15}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

$$\frac{1}{\pi}\cdot\frac{\pi^5}{15} = 0 + \sum_{\substack{n=1 \\ (\text{odd } n)}}^{\infty} 0^2 + \left(\frac{8}{n^3\pi}\right)^2$$

$$\frac{\pi^4}{15} = \frac{64}{\pi^2}\sum_{\substack{n=1 \\ (\text{odd } n)}}^{\infty} \frac{1}{n^6} \Rightarrow \sum_{\substack{n=1 \\ (\text{odd } n)}}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{15\times 64} = \frac{\pi^6}{960}$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \cdots = \frac{\pi^6}{960}$$

O : $f(x) = O(g(x))$?

$$|f(x)| \leq M\,|g(x)|$$

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases} \Longrightarrow b_n = O\left(\frac{1}{n}\right)$$

ناپیوسته

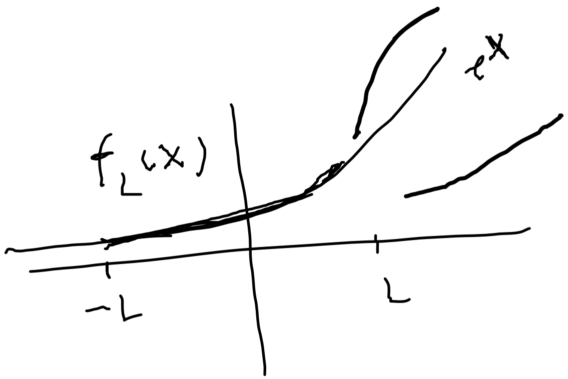
$$f(x) = \begin{cases} x(x+\pi) & -\pi < x < 0 \\ x(-x+\pi) & 0 < x < \pi \end{cases} \Longrightarrow b_n = O\left(\frac{1}{n^3}\right)$$

f پیوسته ، f' پیوسته ، f'' ناپیوسته

f پیوسته, f' ناپیوسته $\Longrightarrow a_n, b_n = O\left(\frac{1}{n^2}\right)$

لم ریمان لبگ :

$$\lim_{\lambda \to \infty} \int f(x) \underset{\downarrow \atop \sin(\lambda x)}{\cos \lambda x}\, dx = 0$$



انتگرال فوریه :

$$\lim_{L\to\infty} f_L(x) \longrightarrow f(x)$$

$$f_L(x) = f(x), \quad -L < x < L$$

$$f_L(x+2L) = f(x)$$

$$f_L(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$L \to \infty$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_L(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_L(x)\sin\frac{n\pi x}{L}\,dx$$

$$f_L(x) = \frac{1}{L}\int_{-L}^{L} f_L(v)\,dv + \sum_{n=1}^{\infty}\Big(\overset{\frac{\Delta w}{\pi}}{\frac{1}{L}}\int_{-L}^{L} f_L(v)\cos\underbrace{\frac{n\pi x}{L}}_{w_n}dv\Big)\cos\underbrace{\frac{n\pi x}{L}}_{w_n}$$

$$+\Big(\underbrace{\frac{1}{L}}_{\frac{\Delta w}{\pi}}\int_{-L}^{L} f_L(v)\sin\underbrace{\frac{n\pi v}{L}}_{w_n}dv\Big)\sin\underbrace{\frac{n\pi x}{L}}_{w_n}$$

$$= \frac{1}{L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Delta w\Big(\int_{-L}^{L} f_L(v)\cos w_n v\,dv\Big)\cos w_n x$$

$$+\Delta w\Big(\int_{-L}^{L} f_L(v)\sin w_n v\,dv\Big)\sin w_n x$$

$w_n = \frac{n\pi}{L}$

$\Delta w = \Delta w_n = w_{n+1} - w_n$

$= \frac{\pi}{L}$

$L \to \infty$

$\Delta w \to 0$

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$$

مطلقاً انتگرال‌پذیر f

$\Delta w \to 0$

$L \to \infty$

$$f(x) = 0 + \frac{1}{\pi}\int_0^{\infty}\Big\{\Big(\underbrace{\int_{-\infty}^{\infty} f(x)\cos wv\,dv}_{A(w)}\Big)\cos wx + \Big(\underbrace{\int_{-\infty}^{\infty} f(v)\sin wv\,dv}_{B(w)}\Big)\sin wx\Big\}\,dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos wx\,dx$$

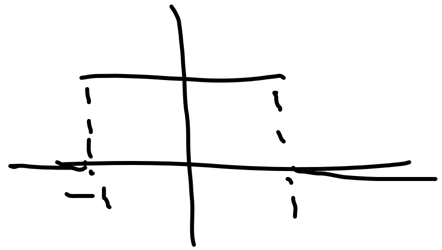
انتگرال فوریه

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin wx\,dx$$

$$f(x) = \int_{0}^{\infty} A(w)\cos wx + B(w)\sin wx\,dw \rightarrow \frac{f(x^+)+f(x^-)}{2}$$

$$f(x) = \begin{cases} 1 & |x|<1 \\ 0 & ,|x|>1 \end{cases}$$

زوج



$$A(w) = \frac{2}{\pi}\int_{0}^{\infty} f(x)\cos wx\,dx = \frac{2}{\pi}\int_{0}^{1} 1.\cos wx\,dx$$

$$= \frac{2}{\pi}\cdot\frac{\sin wx}{w}\Bigg|_{x=0}^{x=1} = \frac{2}{\pi}\cdot\frac{\sin(w)}{w} \longrightarrow \operatorname{sinc}(w) = \frac{\sin w}{w}$$

$B(w) = 0 \longleftarrow$ زوج f

$$\int_0^\infty \left\{ \frac{2}{\pi} \cdot \frac{\sin w}{w} \cdot \cos wx + 0 \cdot \sin wx \right\} dw = \begin{cases} 1 & |x|<1 \\ 1/2 & x=\pm 1 \\ 0 & |x|>1 \end{cases}$$

$$\int_0^\infty \frac{\sin w}{w} \cos wx \, dw = \begin{cases} \pi/2 & |x|<1 \\ \pi/4 & x=\pm 1 \\ 0 & |x|>1 \end{cases}$$

$$\downarrow$$

$$x=0$$

$$\int_0^\infty \frac{\sin w}{w} dw = \frac{\pi}{2}$$

---

f زوج : $B(w)=0$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(x) \cos wx \, dx$$

$$f(x) = \int_0^\infty A(w) \cos wx \, dw$$

f فرد : $A(w)=0$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(x) \sin wx \, dx$$

$$f(x) = \int_0^\infty B(w) \sin wx \, dw$$

مثال. انتگرال فوریه سینوسی و کسینوسی تابع $f(x)=e^{-kx}$ را بدست آورید.
$(k>0)$

① کسینوسی:

$$A(w)=\frac{2}{\pi}\int_0^\infty e^{-kx}\cos wx\,dx$$

$$=\frac{2}{\pi}\left\{-\frac{k}{k^2+w^2}e^{-kx}\left\{-\frac{w}{k}\sin wx+\cos wx\right\}\right\}\Bigg|_0^\infty \quad x$$

$$=\frac{2}{\pi}\cdot\frac{k}{k^2+w^2}$$

$$\int_0^\infty A(w)\cos wx\,dw=\int_0^\infty \frac{2k/\pi}{k^2+w^2}\cos wx\,dw=e^{-kx}$$

② $$\int_0^\infty \frac{\cos wx}{k^2+w^2}\,dw=\frac{\pi}{2k}e^{-kx}\ ;\ (k>0\ ,\ x>0)$$

$$B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin wx\,dx = \frac{2}{\pi}\int_0^\infty e^{-kx}\sin wx\,dx$$

$$= \frac{2w/\pi}{k^2+w^2}$$

$$\int_0^\infty B(w)\sin wx\,dw = \int_0^\infty \frac{2w/\pi}{k^2+w^2}\sin wx\,dw = e^{-kx}$$

$$\int_0^\infty \frac{w}{k^2+w^2}\sin wx\,dw = \frac{\pi}{2}e^{-kx} \qquad (2)$$

(1) و (2) را انتگرالهای لاپلاس می‌گویند.

$$f(x)=\int_0^\infty A(w)\cos wx\,dw \longleftrightarrow A(w)=\frac{2}{\pi}\int_0^\infty f(x)\cos wx\,dx$$

$$A(w) \rightarrow \sqrt{\frac{2}{\pi}}\,\hat{f}_c(w)$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\,dw \longleftrightarrow \hat{f}_c(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\,dx$$

تبدیل عکس کسینوسی

تبدیل فوریه کسینوسی

$$F_c[f]=\hat{f}_c(w) \rightleftarrows f(x)=F_c^{-1}[\hat{f}_c]$$

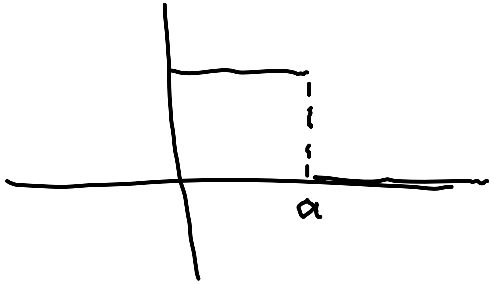
$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)\sin wx\,dw \longleftarrow F_s^{-1}[\hat{f}_s]$$

$$\hat{f}_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx \longleftarrow F_s[f]$$

مثال- تبدیل فوریه سینوسی و کسینوسی تابع

$$f(x)=\begin{cases} k & 0<x<a \\ 0 & x>a \end{cases}$$

k ، a مقدار ثابت



$$F_s\{f\}=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin\omega x\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{a} k\cdot\sin\omega x\,dx=\sqrt{\frac{2}{\pi}}\cdot k\cdot\left.\frac{-\cos\omega x}{\omega}\right|_{x=0}^{x=a}$$

$$=\sqrt{\frac{2}{\pi}}\cdot k\cdot\frac{-\cos(a\omega)+1}{\omega}=\sqrt{\frac{2}{\pi}}\,k\left(\frac{1-\cos a\omega}{\omega}\right)$$

$$F_c\{f\}=\sqrt{\frac{2}{\pi}}\int_0^{a} k\cdot\cos\omega x\,dx=\sqrt{\frac{2}{\pi}}\cdot k\cdot\left.\frac{\sin\omega x}{\omega}\right|_{x=0}^{x=a}$$

$$=\sqrt{\frac{2}{\pi}}\,k\cdot\frac{\sin(a\omega)}{\omega}$$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \; \frac{e^{-x}}{1+w^2} \left(-\cos wx + w \sin wx\right) \Big|_{x=0}^{x=\infty}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+w^2}$$

$$f(x) = k$$

$$0 < x < \infty$$

$$\int_0^\infty |f| \, dx = \int_0^\infty k \, dx = \infty$$

f تبدیل فوریه کسینوسی و سینوسی ندارد

ثابت

$$F_c[af+bg] = \sqrt{\frac{2}{\pi}} \int_0^\infty (af+bg) \cos wx \, dx$$

اعداد

$$= \underbrace{a\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx}_{} + \underbrace{b\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos wx \, dx}_{}$$

$$= a F_c[f] + b F_c[g]$$

F_c یک تبدیل خطی است.

$$F_s[af+bg] = a F_s[f] + b F_s[g]$$

$$F_c[f'] = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{f'(x)}_{du} \underbrace{\cos wx}_{v}\, dx = \sqrt{\frac{2}{\pi}} \left\{ f(x)\cos wx \Big|_0^\infty - \int_0^\infty f(x)(-w \sin wx)\, dx \right\}$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + w \underbrace{\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx\, dx}$$

$$\begin{cases} f(x) \longrightarrow 0 \\ x \longrightarrow \infty \end{cases}$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + w F_s[f]$$

$$F_s[f'] = -w F_c[f]$$

$$F_c[f''] = -\sqrt{\frac{2}{\pi}} f'(0) + w F_s[f'] = -\sqrt{\frac{2}{\pi}} f'(0) - w^2 F_c[f]$$

$$F_s[f''] = -w F_c[f'] = w\sqrt{\frac{2}{\pi}} f(0) - w^2 F_s[f]$$

مثال. تبدیل $F_c[e^{-ax}]$ را بدون انتگرال گیری محاسبه کنید.

$(a>0)$

$$f(x) = e^{-ax} \Rightarrow f'(x) = -a e^{-ax} \longrightarrow f''(x) = a^2 e^{-ax}$$

$$f''(x) = a^2 f(x) \xrightarrow{F_c} F_c[f''] = a^2 F_c[f]$$

$$-\sqrt{\frac{2}{\pi}} f'(0) - w^2 F_c[f] = a^2 F_c[f] \qquad f'(0) = -a$$

$$\sqrt{\frac{2}{\pi}} a = (w^2 + a^2) F_c[f] \Rightarrow F_c[f] = \frac{\sqrt{2/\pi}\, a}{w^2 + a^2}$$

$$F_c^{-1}\left[\frac{\sqrt{\frac{2}{\pi}} a}{w^2 + a^2}\right] = e^{-ax} \quad , \quad x > 0 \, , \, a > 0$$

$$\Downarrow$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sqrt{\frac{2}{\pi}} a}{w^2 + a^2} \cos wx \, dw = e^{-ax} \Rightarrow \int_0^\infty \frac{\cos wx}{w^2 + a^2} dw = \frac{\pi}{2a} e^{-ax}$$

تبدیل فوریه:

$$f(x) = \int_0^\infty A(w)\cos wx + B(w)\sin wx \, dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv \, dv$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv \, dv$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\{\underbrace{\cos wx \cos wv + \sin wx \sin wv}\} dv \, dw$$

$$= \frac{1}{\pi}\int_0^\infty \underbrace{\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\, dv}_{\text{زوج}} \, dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\, dv \, dw + \underbrace{\frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\, dv \, dw}_{0}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\{\cos(wx - wv) + i\sin(wx - wv)\} dv \, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{i(wx - wv)}\, dv\, dw$$

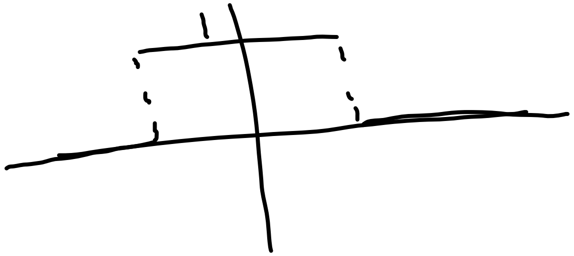
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{-iwv}\, dv \right) e^{iwx}\, dw = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

$$F[f] = \hat{f}(w)$$

$$\begin{cases} f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw & \text{تبدیل فوریه معکوس} \\[2ex] \hat{f}(w) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx & \text{تبدیل فوریه تابع } f : \end{cases}$$

$$F[f] \qquad F^{-1}[\hat{f}]$$

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$



$$\sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

$$F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-iwx}}{-iw} \Bigg|_{x=-1}^{x=1}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-iw} - e^{iw}}{-iw}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{-2i \sin(w)}{-iw}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(w)}{w}$$

$$f(x) = \begin{cases} e^{-ax} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$a > 0$$

$$F[f] = ?$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(a+iw)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-(a+iw)x}}{-(a+iw)} \Bigg|_{x=0}^{x=\infty} = 0 + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a+iw}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{a+iw}$$

$$e^{-(a+iw)x} = e^{-ax} e^{-iwx} = e^{-ax}(\cos wx - i \sin wx)$$

$$F[f'(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{f'(x)}_{du} \underbrace{e^{-iwx}}_{v} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} f(x) e^{-iwx} \Big|_{x=-\infty}^{x=\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(-iwe^{-iwx}\right) dx$$

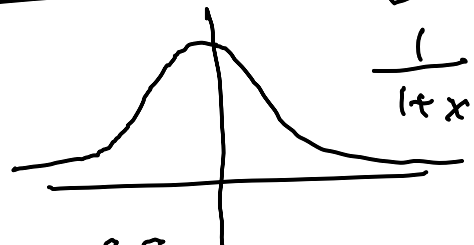
$$= 0 + iwF[f]$$

$$e^{-iwx} = \cos wx - i \sin wx \qquad f(x) \longrightarrow 0, \quad |x| \longrightarrow +\infty$$

e^{-x^2}

$\frac{1}{1+x^2}$

$\vdots$



$$F[f'] = iwF[f]$$

$$F[f''] = iwF[f'] = (iw)^2 F[f] = -w^2 F[f]$$

$$F[f^{(n)}] = (iw)^n F[f]$$

$$F[x e^{-x^2}] = ?$$

$$x e^{-x^2} = \left(-\frac{1}{2} e^{-x^2}\right)'$$

$$F[x e^{-x^2}] = -\frac{1}{2} F\left[\left(e^{-x^2}\right)'\right]$$

$$= -\frac{1}{2} i w F\left[e^{-x^2}\right]$$

$$= -\frac{1}{2} i w \frac{e^{-w^2/4}}{\sqrt{2}} = -\frac{iw}{2\sqrt{2}} e^{-w^2/4}$$

$$F[e^{-ax^2}] = \frac{e^{-w^2/4a}}{\sqrt{2a}}$$

$$a = 1$$

$$F[e^{-ax^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2 - iwx}\, dx$$

$$-ax^2 - iwx = -a\left(x^2 + \frac{iw}{a}x + \left(\frac{iw}{2a}\right)^2 - \left(\frac{iw}{2a}\right)^2\right)$$

$$= -a\left(x + \frac{iw}{a}\right)^2 - w^2/4a$$

$$F\left[e^{-ax^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{iw}{2a}\right)^2 - w^2/4a}\, dx$$

$$= \frac{e^{-w^2/4a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x + iw/2a\right)^2}\, dx$$

$$= \frac{e^{-w^2/4a}}{\sqrt{2a}}$$

$$I = \int_{-\infty}^{\infty} e^{-ax^2}\, dx = ?$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-ay^2} dy\right)$$

$$F[f*g] = \sqrt{2\pi}\, F[f]\, F[g]$$

$f*g \rightarrow$ convolution

تلفیق

$$(f*g)(x) = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp = \int_{-\infty}^{\infty} f(x-p)\, g(p)\, dp$$